Ch: Set & Seq
part 3

Absolute value

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

properties :

(1) $|x| = \max\{x, -x\}$

(2) $-|x| = \min\{x, -x\}$

(3) $|x|^2 = x^2$

(4) $|xy| = |x||y|$

(5) $|x+y| \leq |x| + |y|$

(6) $|x-y| \geq \big||x| - |y|\big|$

Hint for (6)

$x = x - y + y$

$|x| \leq |x-y| + |y|$

$\Rightarrow |x| - |y| \leq |x-y|$

Similarly, (H.W.)

$|y| - |x| \leq |x-y|$

H.W. ☆ Imp

(1) Show that $\max\{x, y\} = \dfrac{x+y+|x-y|}{2}$

$$\min\{x - y\} = \frac{x+y-|x-y|}{2}$$

Sets in $\mathbb{R}$ (continued)

(1) Derived set : The set of all limit pt of a set S is said to be its derived set and denoted by S'.

eg: If $S = (1,2)$, $S' = [1,2]$

If $S = [1,2]$, $S' = [1,2]$

(2) Closed set : If it contains all its limit pt

ie, if $S' \subseteq S$ then S is closed. $\sqrt{2}, \sqrt{3}$

H.W (1) Show $\mathbb{N}$ is closed. (2) $\mathbb{Q}$ is neither open nor closed

HW (1) $\mathbb{R}$ is both open & closed
(2) ϕ " " " " "

(3) A set is open (close) $\Rightarrow$ its complement is close (open)

Dense set: A set S is said to be dense in B if $B \subset S'$

A set S is said to be dense in itself if $S \subset S'$.

ex: S is dense in $\mathbb{R}$ if $S' = \mathbb{R}$
Hence $\mathbb{Q}$ is dense in $\mathbb{R}$.

v. Imp

Nested Interval Theorem

Consider $[a, b]$. If $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a family of CLOSED & BOUNDED nested intervals then $\exists$ one and only one point x such that

$$x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$$

ex: Consider $\left[-\frac{1}{n}, \frac{1}{n}\right]$ for $n = 1, 2, 3, \ldots\ldots$

$$0 \in \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$$

$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) \longrightarrow$ no point

H.W. (1) Show that, finite union/intersection of open sets is open

[Hint: $S_1, S_2, \ldots, S_n \to$ individually open sets

$x \in S_i \Rightarrow \exists\, N_i(x) \subset S_i$, Take ~~intersection~~ intersection of finite neighbourhood $\bigcap N_i(x) \subset \bigcap S_i$

(2) Show that 0 is a limit pt of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

[Hint: Let $\varepsilon > 0$. $\exists\, n \in \mathbb{N} \ni -\varepsilon < \frac{1}{n} < \varepsilon$

$\Rightarrow \frac{1}{n} \in N'(0, \varepsilon) \Rightarrow 0$ is a limit pt of $\left\{\frac{1}{n}\right\}$

(3) Verify Bolzano-Weierstrass theorem for the following sets in $\mathbb{R}$

Just show set is bdd, infinite

(a) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ (b) $\left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$

(c) $\left\{1 + \frac{(-1)^n}{n^2} : n \in \mathbb{N}\right\}$

H.W: (1) Verify whether the following sets are open / close / none

(a) $\{x \in \mathbb{R} : 2x^2 - 5x + 2 < 0\}$

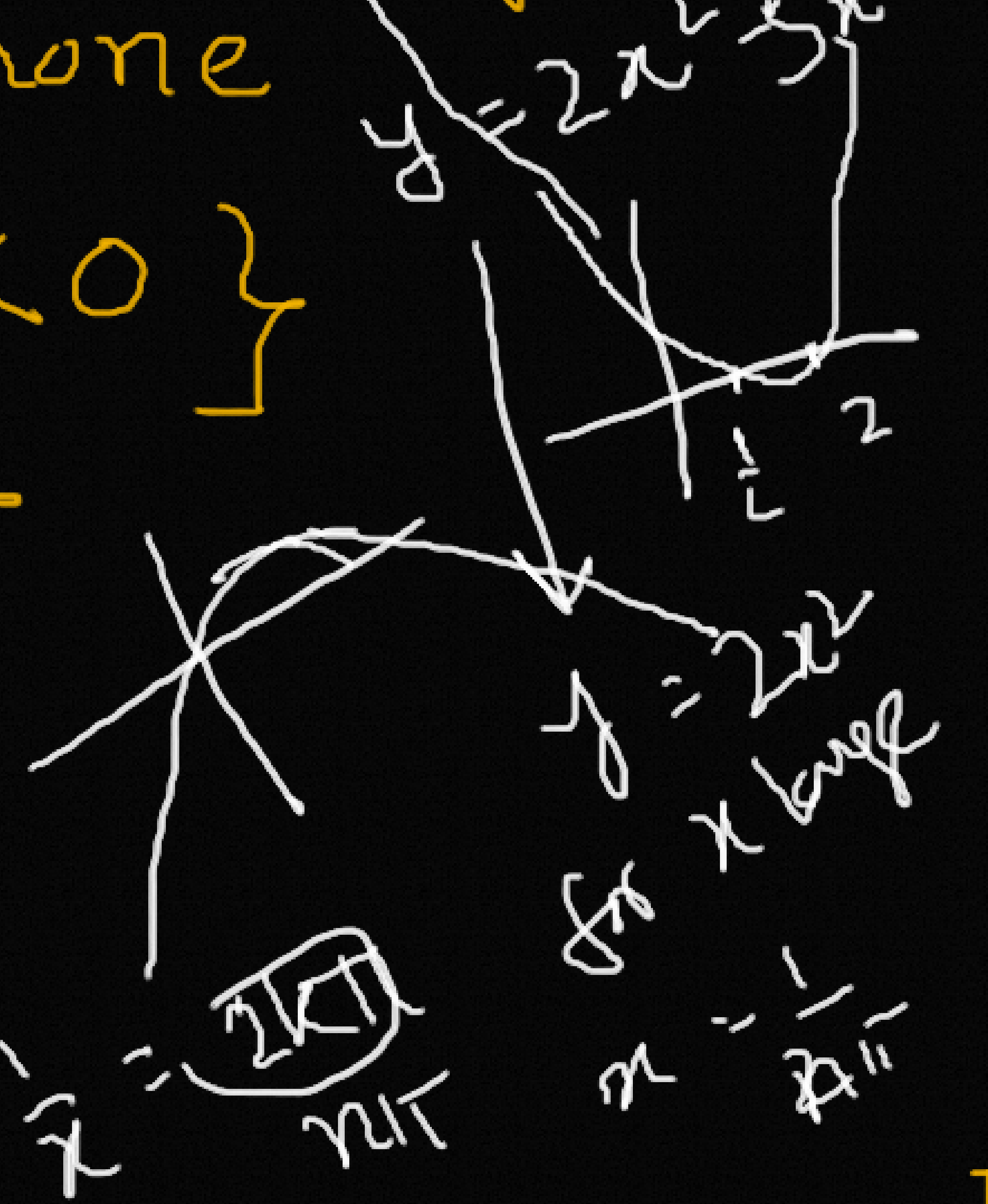
(b) $\{x \in \mathbb{R} : \cos(x) \neq 0\}$

(c) $\{(0,1) \setminus \frac{1}{2^n},\ n \in \mathbb{N}\}$

(d) $\{x \in \mathbb{R} : \sin x = 0\}$

(e) $\{x \in \mathbb{R} : \sin \frac{1}{x} = 0\}$

(f) $\bigcup_{n=1}^{\infty} I_n$ where $I_n = \{x \in \mathbb{R} : \frac{1}{4^n} \leq x \leq \frac{1}{2}\}$



H.W

$\mathbb{N}$

(1) Prove that a finite subset of $\mathbb{R}$ is close set

(2) Show that finite union/intersection of close set is close. $A_1, A_2, \ldots, A_n \rightarrow$ closed

De Morgan

$\bigcap B(x, r_i)$

$\bigcup^n A_i$

$C A_i$ $\quad A_i^c$

$\left(\bigcup A_i\right)^c = \left(\bigcap A_i^c\right)$

$x \in A_i^c$

$B(x, r_i)$

enough to show

is open

x

Borel Set

A set that can be obtained from the union, intersection and relative complement of enumerable collection of close & open sets in $\mathbb{R}$ is known as Borel set.

eg; (V. Imp example of Borel set) Borel set generated by $[a,b]$, $[a,b)$, $(a,b]$, (a,b)

$a, b \in \mathbb{R}$

Cover

Let $\mathbb{C}$ be an arbitrary collection of sets $\{A_\alpha : \alpha \in \Lambda\}$. $\mathbb{C}$ is said to be a cover of $S \, (\subset \mathbb{R})$ if

$$S \subset \bigcup_{\alpha \in \Lambda} A_\alpha$$

✱ If $\{A_\alpha : \alpha \in \Lambda\}$ all open $\Rightarrow$ open cover

✱ If $\mathbb{C}$ is a cover of S and a subcollection $\mathbb{C}'$ of $\mathbb{C}$ is also cover of S then $\mathbb{C}'$ = **Sub Cover**

Heine-Borel Theorem

Let $S (\subset \mathbb{R})$ is closed and bounded. Then every open cover of S has a finite sub cover.

V.Imp

Compact set

$S (\subset \mathbb{R})$ is compact set if every open cover of S has a finite subcover.

Hence, a closed and bounded set in $\mathbb{R}$ is compact

SEQUENCE

A sequence is a mapping $f : \mathbb{N} \to \mathbb{R}$ and denoted by $\{f(n)\}_{n \in \mathbb{N}}$ or $\{x_n\}_{n \in \mathbb{N}}$

Here $f(n)$ or x_n = n th term of seq

ex: (1) $x_n = n^2 \Rightarrow$ infinite seq of squares

(2) $x_n = \frac{1}{n(n-1)} \Rightarrow \{x_n\}$ is a seq for $n \geq 2$

(3) $x_n = \tan\left(\frac{n\pi}{2}\right)$ is NOT a seq (not defined for infinitely many n)

Bounds of sequence $\{x_n\}$

* $\{x_n\}$ bdd above if $\exists\, b \in \mathbb{R} \ni x_n \leq b \ \forall n \in \mathbb{N}$
* $\{x_n\}$ bdd below if $\exists\, a \in \mathbb{R} \ni x_n \geq a \ \forall n \in \mathbb{N}$

a = Lower bound , b = upper bound

* $\{x_n\}$ is bdd if both a, b exists

eg:- $\{\frac{1}{n}\}$ bdd below with supremum 1, infimum 0

$\{(-1)^n\}$ bdd seq (inf -1 , sup 1)

H.W: Find sup, inf of $\{x_n\} = \left\{(-1)^n \frac{n^2}{n^2+2}\right\}$ [ans: -1 & 1]

Convergence of a sequence

A seq $\{x_n\}$ is said to converge to a real number l (or, equivalently, we say $\{x_n\}$ has limit l) iff for any given $\epsilon > 0$, $\exists N \in \mathbb{N} \ni |x_n - l| < \epsilon \; \forall n \geq n_0$

$\exists$ = there exists
$\in$ = inclusion
$\ni$ = such that
$\forall$ = for all

If $\{x_n\}$ converges to l we write

$\lim_{n \to \infty} x_n = l$ or $x_n \to l$

Ex (1) Show that, $\{\frac{1}{n}\} \to 0$

Here $\left|\frac{1}{n} - 0\right| = \frac{1}{n}$, for any $\varepsilon > 0$,

$\left|\frac{1}{n} - 0\right| < \varepsilon$ if $n > \frac{1}{\varepsilon}$. Take $N = [\frac{1}{\varepsilon}] + 1$

$\Rightarrow \exists\ N \in \mathbb{N} \ni \left|\frac{1}{n} - 0\right| < \varepsilon\ \forall\ n \geq N$ (proved)

(2) Show that $\{x^n\} \to 0$ if $|x| < 1$

Here $\left|\frac{1}{x}\right| > 1 \Rightarrow \frac{1}{|x|} = 1 + h$ for some $h > 0$

Now $|x^n - 0| = \frac{1}{(1+h)^n} = \frac{1}{1 + nh + \binom{n}{2}h^2 + \cdots + h^n} < \frac{1}{nh}$

So, $\forall \varepsilon > 0$, $|x^n - 0| < \varepsilon$ if $n > \frac{1}{h\varepsilon}$ [Take $N = [\frac{1}{nh}] + 1$

③ Show that :

(i) $\left\{ \frac{1}{n^p} \right\}$ converges to 0 if $p > 0$

$(5n+1)^{\frac{1}{4n}}$

$(\star+1)^{\frac{1}{\star}}$

$\frac{5}{n} = 1$

(ii) $\left\{ (n+1)^{\frac{1}{n}} \right\}$ converges to 1

$(\star+1)^{\frac{1}{\star}} \longrightarrow 1$ as $\star \to \infty$

(iii) $\frac{n}{n+1} \longrightarrow 1$

$\left(1+\frac{1}{n}\right)^n \longrightarrow e$

(iv) $\frac{n^2}{n^2+1} \longrightarrow 1$

$\left(1+\frac{1}{5n}\right)^{4n} \longrightarrow e^{\frac{4}{5}}$

(v) Show that, a constant seq is conv.

$\lim a^{\lim b} = (\lim a)^{\lim b}$

(4) Show that, $\{n^{\frac{1}{n}}\}$ conv. to 1

$(1+(n-1))^{1/n} \to \lim(1+(n-1))^{\frac{1}{n-1} \cdot \frac{n-1}{n}} = 1$

$\lim \left(1+(n-1)\right)^{\frac{1}{n-1}\left(\frac{n-1}{n}\right)}$

Here, $n^{\frac{1}{n}} > 1$ for $n > 1$

$\Rightarrow n^{\frac{1}{n}} = 1 + x_n$ (here $x_n > 0$ for $n > 1$)

We will show $x_n \to 0$

$$(1+x_n)^n = n$$

$$\Rightarrow n = 1 + n x_n + \binom{n}{2} x_n^2 + \binom{n}{3} x_n^3 + \cdots + x_n^n$$

$$\Rightarrow n > \frac{n(n-1)}{2} x_n^2 \Rightarrow x_n < \sqrt{\frac{2}{n-1}}$$

Take N accordingly

$$\Rightarrow \forall \varepsilon > 0, |x_n - 0| < \varepsilon \text{ if } \sqrt{\frac{2}{n-1}} > \varepsilon \text{ ie } n > \left[1 + \frac{2}{\varepsilon^2}\right] + 1$$

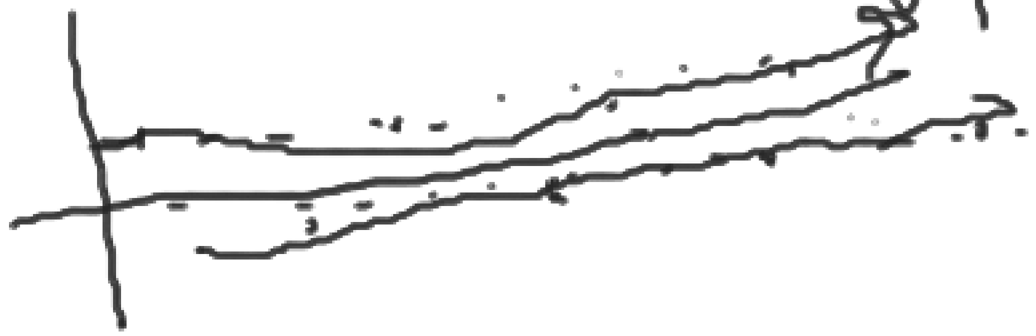
Divergent and oscillatory seq

$\{x_n\}$ is said to diverge to ∞ if corresponding to any preassigned positive number M, $\exists\ k \in \mathbb{N} \ni x_n > M \ \forall\ n \geq k$.

$\{x_n\}$ diverge to $\infty \Rightarrow \lim\limits_{n \to \infty} x_n = \infty$

* Not convergent $\not\Rightarrow$ divergent.

This type of seq called oscillatory



$(-1)^n = -1, 1, -1, 1,$

Oscillate finitely

if $\exists\ \alpha \ni |x_n| < \alpha \ \forall\ n \in \mathbb{N}$

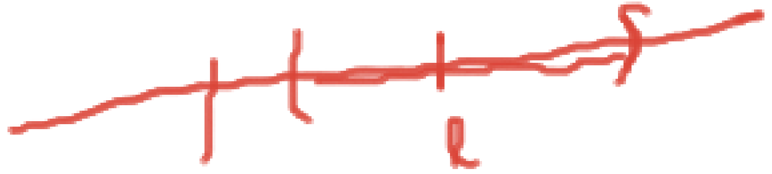
otherwise, it is said to oscillate infinitely.

H.W. Show that $\{n^2\} \to \infty$ $\left[\begin{array}{l} n^2 > M \Rightarrow n > \sqrt{M} \\ \text{Take } \ k = [\sqrt{M}] + 1 \end{array}\right]$

$\{(-1)^n\}$ oscillates finitely

$\{(-1)^n n\}$ oscillates infinitely

Theorem convergent seq has unique limit.



proof: Let $\{x_n\}$ has two limits l_1, l_2.

Consider $\varepsilon = \frac{1}{2}|l_1 - l_2| > 0$ such that $\left[\text{ie}, \; |l_1 - l_2| = 2\varepsilon \right]$

$\{x_n\} \to l_1 \Rightarrow \forall \varepsilon > 0, \exists n_1 \ni |x_n - l_1| < \varepsilon \;\; \forall n \geq n_1$

$\{x_n\} \to l_2 \Rightarrow \forall \varepsilon > 0, \exists n_2 \ni |x_n - l_2| < \varepsilon \;\; \forall n \geq n_2$

Now, $|l_1 - l_2| = |l_1 - x_n + x_n - l_2| \leq |x_n - l_1| + |x_n - l_2|$

$< 2\varepsilon \;\; \forall n \geq n_3$

Take $n_3 = \max(n_1, n_2) \Rightarrow |l_1 - l_2| < |l_1 - l_2| \Rightarrow\Leftarrow$ (Contradicti